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LETTER TO THE EDITOR

Self-Fourier functions and coherent laser combination

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Abstract
The Gaussian and Comb functions are generally quoted as being the two basic functions that are their own Fourier transforms. In 1991, Caola presented a recipe for generating functions that are their own Fourier transforms by symmetrizing any transformable function and then adding its own Fourier transform to it. In this letter, we present a new method for generating a set of functions that are exactly their own Fourier transforms, and which have direct application to laser cavity design for a wide variety of applications. The generated set includes the Gaussian and Comb functions as special cases and forms a continuous bridge of functions between them. The new generating method uses the Gaussian and Comb functions as bases and does not rely on the Fourier operator itself. This self-Fourier function promises to be particularly useful in high-power laser design through coherent laser beam combination. Although these results are presented in a single dimension as with a linear array, the results are equally valid in two dimensions.

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Background

As can be seen in the literature [1], the Gaussian and Comb functions are often stated as being prime examples of functions that are their own Fourier transform, or self-Fourier functions (SFFs). Specifically, the Gaussian function $A(x)$ with Gaussian width a

$$A(x) = \exp[-(x/a)^2]$$

is its own transform if a is equal to $1/\sqrt{\pi}$.

Likewise, the Comb function $B(x)$ with spacing b

$$B(x) = \sum_{n=-\infty}^{\infty} \delta(x - n \cdot b)$$

is its own transform if b is equal to 1.0.

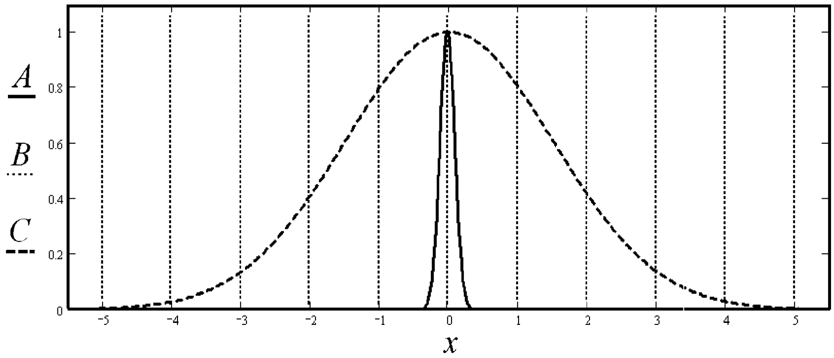

Figure 1. Three basis functions: narrow Gaussian A, Comb B and wide Gaussian C.

Self-Fourier functions have been discussed many times in the literature [1–8], and have been pointed out to be relevant and useful in optical cavity design.

In 1991, Caola [1] describes a set of arbitrary functions that can be generated which are their own transforms.

In 1992, Lohmann [2] asserts that Caola discovered all SFFs. We point out, however, that Caola did not necessarily discover all SFFs. He discovered a subjective mapping from an arbitrary function $g(x)$ to a corresponding SFF.

In 1994, Liu [6] presents a recipe for generating a series of approximate self-transforms using the same equation structure described here.

Bracewell [7] uses Gaussians in the same equation structure, also yielding an approximate self-transform function.

The method described here is exact, however, due to the very special nature of the Gaussian function used as both the 'cell function' and the 'aperture function' if the function parameters are adjusted slightly.

The SFFs generated using the proposed method are directly applicable to coherent laser combination as detailed later in this letter. In addition, they are capable of being precisely tailored to specific designs for high output power lasers. In fact, this SFF is the basis of a new phase-locked laser array being designed and demonstrated for high-power applications.

Self-Fourier function generation

In this letter, we describe a new method for generating a set of SFFs based on the Gaussian and Comb functions. The SFFs generated by this new method are directly applicable to the coherent operation of laser arrays.

This new function $F(x)$ is generated using three basis functions. Two of these are Gaussian functions: $A(x)$ and $C(x)$ with Gaussian widths of a and c, respectively. The third is a Comb function: $B(x)$ with spacing b. These three basis functions are presented in figure 1.

For the examples of figures 1 and 2, the following basis function parameters were chosen:

$$a \approx 0.15 \tag{1}$$

$$b \approx 1.00 \tag{2}$$

$$c \approx 2.12. \tag{3}$$

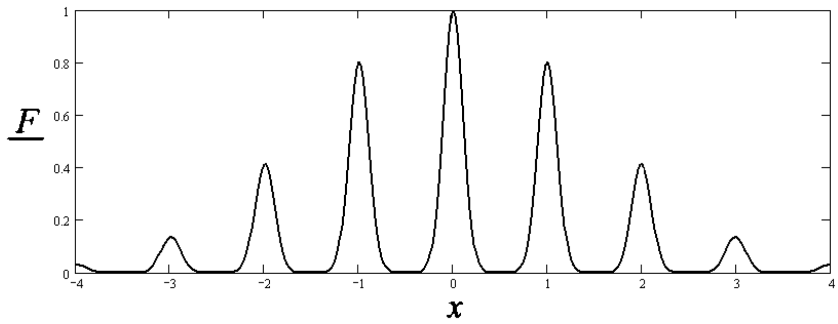

Figure 2. Total self-Fourier function $F(x)$ formed by multiplying a wide Gaussian by a Comb then convolving with a narrow Gaussian.

These three functions are combined in the following manner (the x-dependent notation is omitted for clarity):

$$F = A \otimes (B \cdot C) \tag{4}$$

where $\otimes$ denotes the convolution operator. Equation (4) can be written as

$$F(x) = \sum_n \{\exp[-([x - (n \cdot b)]/a)^2] \exp[-(n \cdot b/c)^2]\}. \tag{5}$$

The function $F(x)$ is presented in figure 2.

As is suggested by the amplitude pattern shown in figure 2, this could describe the output from a linear (or a two-dimensional) array of laser elements with a Gaussian width of 0.15 with a spacing of 1.0 (both parameters are dimensionless for the purpose of this letter).

The values of a, b and c in the total function can be adjusted to obtain an infinite set of SFFs of the form shown. We will see later that there will be only one free parameter. This free parameter can be varied smoothly to obtain a continuous set of SFFs including the basic Gaussian and Comb functions.

The parameters $\{a, b, c\}$ are determined as follows.

It can be shown (quite remarkably) that the composite function presented in equation (4) is, in fact, equal to that presented in equation (6) with appropriate choice of parameters:

$$F = (A' \otimes B') \cdot C' \tag{6}$$

or

$$F(x) = \left\{ \sum_n \exp[-([x - (n \cdot b')]/a')^2] \right\} \exp[-(x/c')^2] \tag{7}$$

if the functions A', B' and C' have their widths and spacings set to a', b' and c', respectively.

In other words, if a linear array of evenly spaced Gaussians with equal widths is multiplied by a Gaussian envelope, the result remains a linear array of evenly spaced Gaussians, all with equal widths. This new array will have an amplitude distribution amongst the peaks which is also exactly Gaussian. The widths and spacing of the new array have been changed from the original, and this shift must be (and can be) accounted for in creating the SFF. This unusual property of the Gaussian function is what permits this self-transform function to be made exact. We believe the Gaussian function to be the only function with this property.

It should also be noted that no heuristic is needed for the ratios of the widths of the Gaussian peaks or envelope as described in [6, 7]. The *exact* nature of the self-Fourier property even holds for overlapped Gaussians.

Equation (6) is closely related to equation (4), in that it involves a narrow Gaussian, a Comb and a wide Gaussian as bases. However, the order of the convolution and multiplication are interchanged.

By comparing the different terms (x and x^2) of equations (5) and (7) (by moving all factors inside of the summation operator and combining exponential terms), and then setting the coefficients of x and x^2 to be equal, we can determine the following two relations:

$$\frac{1}{a^2} = \frac{1}{a'^2} + \frac{1}{c'^2} \tag{8}$$

and,

$$\frac{b}{b'} = \frac{a^2}{a'^2}. \tag{9}$$

That is, the effective widths a' of the narrow Gaussians $A'(x)$ appearing in the total function are slightly modified by being multiplied by the Gaussian envelope $C'(x)$ in equation (6). The spacings of the peaks are also slightly modified as well as the width of the envelope.

Now, using standard Fourier transform theory, we can take the Fourier transform of equation (4) to obtain

$$\mathrm{FT}\{A \otimes (B \cdot C)\} = \mathrm{FT}(A) \cdot \{\mathrm{FT}(B \cdot C)\} \tag{10}$$

or

$$\mathrm{FT}\{A \otimes (B \cdot C)\} = \mathrm{FT}(A) \cdot \{\mathrm{FT}(B) \otimes \mathrm{FT}(C)\}. \tag{11}$$

Setting this transform equal to itself, we obtain

$$\mathrm{FT}(A) \cdot \{\mathrm{FT}(B) \otimes \mathrm{FT}(C)\} = A \otimes \{B \cdot C\} \tag{12}$$

or, using equation (6), we obtain

$$\mathrm{FT}(A) \cdot \{\mathrm{FT}(B) \otimes \mathrm{FT}(C)\} = (A' \otimes B') \cdot C'. \tag{13}$$

Using the commutative property, we obtain

$$\mathrm{FT}(A) \cdot \{\mathrm{FT}(B) \otimes \mathrm{FT}(C)\} = C' \cdot (B' \otimes A'). \tag{14}$$

The right- and left-hand sides of equation (14) will be equal if we set

$$\mathrm{FT}(A) = C' \tag{15}$$

$$\mathrm{FT}(B) = B' \tag{16}$$

$$\mathrm{FT}(C) = A'. \tag{17}$$

Now, since $A(x)$, $C(x)$, $A'(x)$, and $C'(x)$ are all Gaussian functions, we can satisfy equations (15) and (17) through appropriate choice of Gaussian widths. Equation (16) is satisfied through appropriate choice of the spacing of the teeth in the Comb function, or

$$a = \frac{1}{\pi \cdot c'} \tag{18}$$

$$b = \frac{1}{b'} \tag{19}$$

$$c = \frac{1}{\pi \cdot a'}. \tag{20}$$

Now, equations (8), (9), (18)–(20) form a set of five equations with six unknowns (a, b, c, a', b' and c') that can be solved simultaneously for the widths and spacings of the basis functions. There is a single degree of freedom left over, which we choose as the narrow

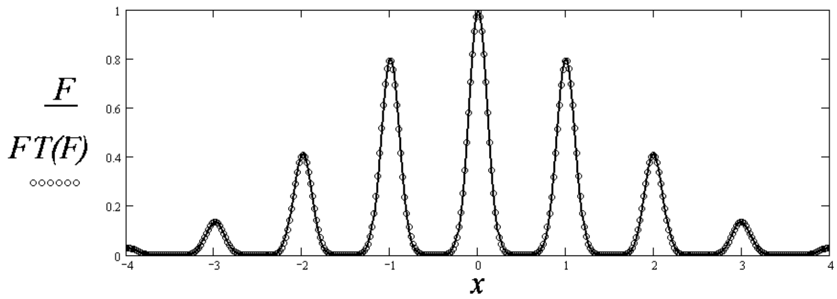

Figure 3. Self-Fourier function along with its Fourier transform.

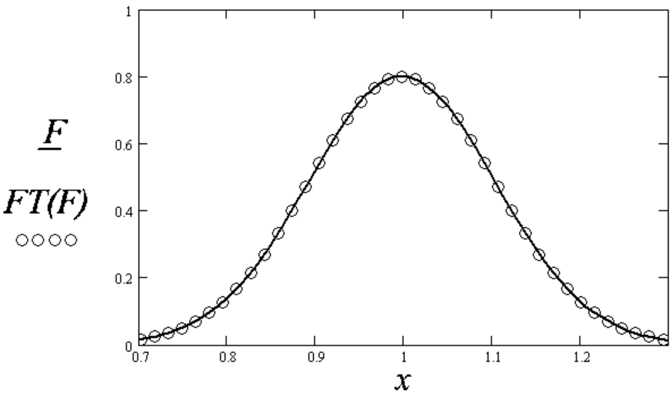

Figure 4. Self-Fourier function with its Fourier transform (close-up of single peak).

Gaussian width a, which we set equal to 0.15 for the example. The choice of a from the set $\{a, b, c, a', b', c'\}$ was arbitrary. We then obtain the following relations for b and c:

$$a = 0.15 \tag{21}$$

$$b = \sqrt{1 - \pi^2 \cdot a^4} \tag{22}$$

$$c = \frac{b}{\pi \cdot a} \tag{23}$$

leaving us with the following set of parameters:

$$a = 0.150\,000\,000 \tag{24}$$

$$b = 0.997\,498\,628 \tag{25}$$

$$c = 2.116\,757\,832 \tag{26}$$

and the complete function

$$F(x) = \sum_n \{\exp[-([x - (n \cdot b)]/a)^2]\ \exp[-(n \cdot b/c)^2]\} \tag{27}$$

being an exact self-Fourier function.

This pattern has been shown previously in figure 2. It is shown again in figure 3 along with its Fourier transform $\mathrm{FT}(F)$.

A close-up of a single peak is presented in figure 4.

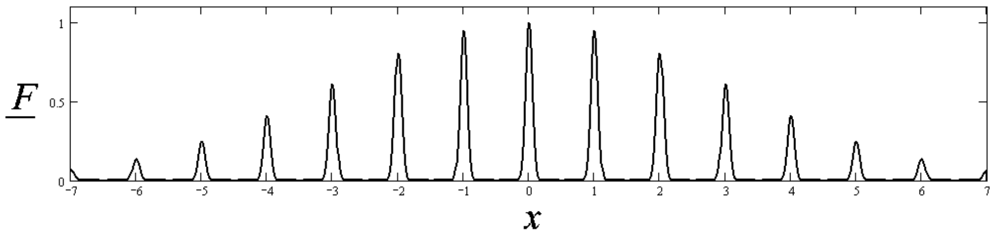

Figure 5. Self-Fourier function with smaller 'a' of 0.075 showing a second member of the continuous set of SFFs generated by the new method.

As mentioned, this function maintains a single degree of freedom (a) which can be varied to increase/decrease the fill factor and the number of peaks to create a continuous family of SFFs. The family of SFFs includes the Dirac Comb and Gaussian as special cases.

In the limit as the parameter a approaches the value of 0.0 (b approaches 1.0 and c approaches ∞), this function approaches the Dirac Comb function, as discussed by Bracewell [7].

In the limit as the parameter a approaches the value of $\pi^{-1/2}$ (b approaches 0.0 and c approaches 0.0), this function approaches the regular Gaussian function with Gaussian width $\pi^{-1/2}$.

In the limit as the parameter a approaches $+\infty$ (b approaches ∞ ($\sim\pi \cdot a^2 \cdot i$) and c also approaches ∞ ($\sim a \cdot i$)), this function again approaches the Dirac Comb function.

It should be noted that as a is increased from $\pi^{-1/2}$ towards ∞, the function reproduces the same patterns as when the value of a is decreased from $\pi^{-1/2}$ towards 0.0, differing only by the scale factor $\pi^{1/2} \cdot a$.

That is, for all real a

$$F(x, a) = (\sqrt{\pi} \cdot a) \cdot F(x, 1/(\pi \cdot a)). \tag{28}$$

It can also be shown that

$$F(x, -a) = F(x, a). \tag{29}$$

Discussion of the form of this SFF with the parameter a being complex will be the subject of a future paper.

It should be noted that a subset of the exact results above were approximated by Bracewell [7] (see figure 7.19, p 257). In particular, Bracewell arrived at the Dirac Comb in the limit as the parameter a approached zero.

The SFF obtained using the value of $a = 0.075$ is presented in figure 5.

As can be seen, the width of these peaks is half the width of those presented in figure 2 and there are twice as many of them.

Coherent laser beam combination

As described in previous articles, SFFs such as presented in this letter are useful in optical design where lenses are used to effect spatial Fourier transforms and it is desirable to reproduce an optical pattern in the Fourier plane. Note, in an optical design using spatial Fourier transforms such as described above, that the three dimensionless parameters a, b, and c are each scaled by the linear distance SQRT($F_L \cdot \lambda$) where F_L is the focal length of the lens and λ is the operating wavelength.

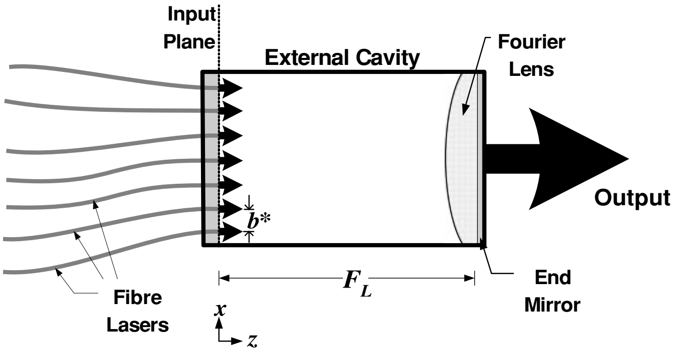

Figure 6. Coherent combination of individual laser elements.

This family of SFFs is especially useful for effecting the coherent beam combination of an array (linear or two-dimensional) of individual laser emitters. One example of such an array is presented below using fibre lasers (although use of many other types of lasers is possible).

Coherent beam combination can be achieved by arranging an array of laser gain elements (lasers with their individual internal feedback removed) in an external cavity as shown in figure 6. The scaled spacing b^* is chosen according to equation (22) and describes the spacing between the fibre inputs. The scaled parameter a^* describes the output beam width of the fibre lasers (approximately Gaussian shaped). The width of the feedback envelope c^* (and thus the number of elements in the array) is determined by equation (23).

By specifically arranging the elements in the input plane to achieve the SFF, the feedback interference pattern achieved back at the input plane will efficiently couple back into the individual fibre lasers. This will force lasing to occur in a collective manner, resulting from the Fourier nature of the optical feedback. Light that is not coherent between the elements suffers extremely high loss and does not contribute to the collective feedback.

Note that in the optical system presented in figure 6, the spatial Fourier transform of the pattern at the input plane is overlaid back on the input plane. In this case, F_L is the *round-trip* focal length of the lens (= 5.84 cm). With an operating wavelength of 1.07 μm, this gives a scaling factor of 250 μm for this design.

In this optical design, the curve presented in figure 3 shows the field amplitude of the array at the input plane to the cavity as a function of the distance along the array axis (x).

As a practical 'rule-of-thumb' for efficient cavity operation, the number of elements along the axis in the array (N) should be chosen according to equation

$$N \sim (b/a) \tag{30}$$

which for this case is approximately 7. This choice of N will result in efficient capture of all of the far field lobes emitted by the coherent array of emitters. As can be seen from this equation (and from figure 5), a larger number of elements in the array is achieved by increasing the ratio of b/a (reducing the fill factor of the array).

Coherent combination of multiple lasers utilizing these same physics and optics principles has been experimentally demonstrated [9–13]. However, in this new design, the input array of fibre lasers performs the role of the spatial filter described in these references. This system has been 'folded' and simplified to provide increased stability, reduction in cavity size and increased ruggedness.

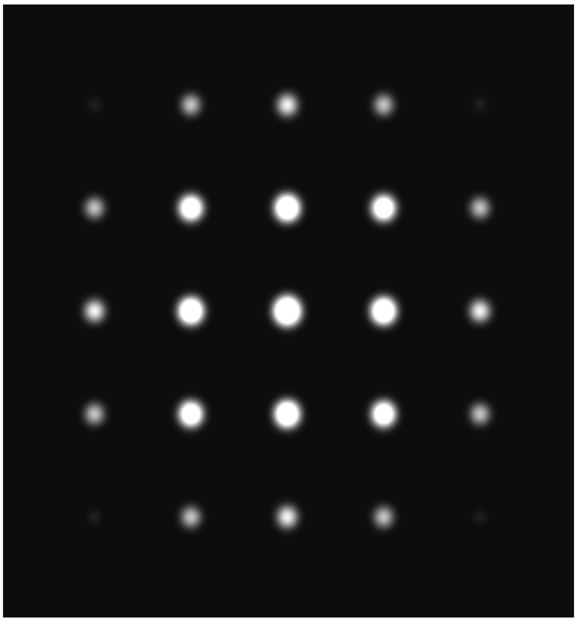

Figure 7. Example of self-Fourier function for 2D laser array.

Note that, if the array being designed is a *linear* array of lasers, an additional cylindrical lens is required to image the light back upon itself in the dimension transverse to the array dimension. This cylindrical lens is not required for a two-dimensional array of laser elements.

Creating a two-dimensional rectilinear array (F_{2D}) which is an SFF is simply achieved through use of equation (5) in two dimensions:

$$F_{2D}(x, y) = F(x) \cdot F(y) \tag{31}$$

where $F(x)$ is defined by equation (5).

One example of such an array is presented in figure 7. Figure 7 presents the intensity of a rectilinear array as a function of position in the x–y plane.

This coherent laser combination technique using SFFs is currently the focus of Phase II SBIR being performed for the U.S. Navy under SBIR program number N00178-03-C-3005 [14, 15]. It is related to the subject matter of United States Patent 6,714,581. Inventor: Corcoran, Christopher J.

Specific applications

The coherent combination of individual laser emitters has many uses in applications where a high-power high-brightness laser beam is required or highly desirable. Use of this technique will offer the advantages of simplicity, robustness and scalability compared to many other combination techniques.

Free space optical communication requires a relatively high-power high-brightness laser beam to be able to transmit over longer distances. Coherent beam combination using this technique offers the ability to scale up the power and transmission range by simply adding a relatively simple element (Fourier cavity) directly onto a multi-element transmitter output.

The industrial sector has many applications for high-power high-brightness laser beams for cutting, welding, drilling or marking various materials. An increase in power and brightness will result in an increase in processing speed, thereby making the process more economic. It will also result in the ability to process new or thicker materials that are too difficult to process with currently available laser sources.

The military is already using high-power high-brightness laser beams for several applications. One application of particular interest is long range illumination and targeting, where several hundred watts of optical power is required with a high beam quality. This technique will allow the use of compact and efficient solid-state lasers providing a much more rugged and reliable system compared to existing laser sources.

Another military application is directed energy. Here again, the combination of a multitude of laser beams using this technique will provide a scalable source capable of delivering as much power as desired by simply increasing the number of laser elements. This technique is directly applicable to almost any type of laser. It could be used with high efficiency lasers to provide a compact and portable directed energy source.

Summary

In summary, we have presented a new method for generating a continuous set of functions that are their own Fourier transforms, and that have direct application to laser cavity design for a wide variety of applications. The new generating method uses the Gaussian and Comb as bases and does not rely on the Fourier operator itself. The generated set includes the Gaussian and Comb functions as special cases and forms a continuous bridge of functions between them. This SFF promises to be particularly useful in high-power laser design through the use of coherent laser beam combination.

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